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# On characteristic points at blow-up for a semilinear wave equation in one space dimension<sup>\*</sup>

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We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_{tt}^2 u = \partial_{xx}^2 u + |u|^{p-1} u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \quad (1)$$

where  $u(t) : x \in \mathbb{R} \rightarrow u(x, t) \in \mathbb{R}$ ,  $p > 1$ ,  $u_0 \in H_{\text{loc},u}^1$  and  $u_1 \in L_{\text{loc},u}^2$  with  $\|v\|_{L_{\text{loc},u}^2}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$  and  $\|v\|_{H_{\text{loc},u}^1}^2 = \|v\|_{L_{\text{loc},u}^2}^2 + \|\nabla v\|_{L_{\text{loc},u}^2}^2$ .

The Cauchy problem for equation (1) in the space  $H_{\text{loc},u}^1 \times L_{\text{loc},u}^2$  follows from the finite speed of propagation and the wellposedness in  $H^1 \times L^2$  (see Ginibre, Soffer and Velo [5]). If the solution is not global in time, then we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques, or also by the following blow-up criterion from Levine [8]:

*If  $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$  satisfies*

$$\int_{\mathbb{R}} \left( \frac{1}{2} |u_1(x)|^2 + \frac{1}{2} |\partial_x u_0(x)|^2 - \frac{1}{p+1} |u_0(x)|^{p+1} \right) dx < 0,$$

*then the solution of (1) cannot be global in time.*

More blow-up results can be found in Caffarelli and Friedman [4], [3], Alinhac [1] and Kichenassamy and Litman [6], [7].

If  $u$  is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve  $\Gamma = \{(x, T(x))\}$  such that  $u$  cannot be extended beyond the set called the maximal influence domain of  $u$ :

$$D = \{(x, t) \mid t < T(x)\}. \quad (2)$$

$\bar{T} = \inf_{x \in \mathbb{R}} T(x)$  and  $\Gamma$  are called the blow-up time and the blow-up graph of  $u$ . A point  $x_0$  is a non characteristic point (or a *regular* point) if

$$\text{there are } \delta_0 \in (0, 1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0, T(x_0), \delta_0} \cap \{t \geq t_0\} \quad (3)$$

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where  $\mathcal{C}_{\bar{x}, \bar{t}, \bar{\delta}} = \{(x, t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}$ . If not, then we call  $x_0$  a characteristic point (or a *singular* point). Naturally, we denote by  $\mathcal{R}$  (resp.  $\mathcal{S}$ ) the set of non characteristic (resp. characteristic) points. Note then that

$$\mathcal{R} \cup \mathcal{S} = \mathbb{R}.$$

In our papers [11], [13], [12], [14], [15] and [16], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

## 1 The blow-up graph of equation (1)

It is clear from rather simple arguments that  $\mathcal{R} \neq \emptyset$  for *any* blow-up solution  $u(x, t)$  (if  $T(x)$  achieves its minimum at  $x_0$ , then  $x_0 \in \mathcal{R}$ ; if the infimum is at infinity, then see the remark following Theorem 1 in [15]). On the contrary, the situation was unclear for  $\mathcal{S}$ , and it was commonly conjectured before our contributions that  $\mathcal{S}$  was empty. In particular, that was the case in the examples constructed by Caffarelli and Friedman in [4] and [3]. In [16], we prove that the conjecture was false. More precisely, we proved the following:

**Proposition 1 (Existence of initial data with  $\mathcal{S} \neq \emptyset$ )** *If the initial data  $(u_0, u_1)$  is odd and  $u(x, t)$  blows up in finite time, then  $0 \in \mathcal{S}$ .*

For general blow-up solutions, we proved the following facts about  $\mathcal{R}$  and  $\mathcal{S}$  in [15] and [16] (see Theorem 1 (and the following remark) in [15], see Propositions 5 and 8 in [16]):

### Theorem 2 (Geometry of the blow-up graph)

- (i)  $\mathcal{R}$  is a non empty open set, and  $x \mapsto T(x)$  is of class  $C^1$  on  $\mathcal{R}$ ;
- (ii)  $\mathcal{S}$  is a closed set with empty interior, and given  $x_0 \in \mathcal{S}$ , if  $0 < |x - x_0| \leq \delta_0$ , then

$$0 < T(x) - T(x_0) + |x - x_0| \leq \frac{C_0|x - x_0|}{|\log(x - x_0)|^{\frac{(k(x_0)-1)(p-1)}{2}}} \quad (4)$$

for some  $\delta_0 > 0$  and  $C_0 > 0$ , where  $k(x_0) \geq 2$  is an integer. In particular,  $T(x)$  is right and left differentiable at  $x_0$ , with  $T'_l(x_0) = 1$  and  $T'_r(x_0) = -1$ .

## 2 Asymptotic behavior near the blow-up graph

As one may guess from the above description, the asymptotic behavior will not be the same on  $\mathcal{R}$  and on  $\mathcal{S}$ . In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some  $(x_0, T_0)$  such that  $0 < T_0 \leq T(x_0)$ , we introduce the following self-similar change of variables:

$$w_{x_0, T_0}(y, s) = (T_0 - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \quad (5)$$

If  $T_0 = T(x_0)$ , then we simply write  $w_{x_0}$  instead of  $w_{x_0, T(x_0)}$ . The function  $w = w_{x_0, T_0}$  satisfies the following equation for all  $y \in B = B(0, 1)$  and  $s \geq -\log T_0$ :

$$\partial_{ss}^2 w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w - \frac{p+3}{p-1}\partial_s w - 2y\partial_{y,s}^2 w \quad (6)$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \quad (7)$$

From Antonini and Merle [2], we know the existence of the following Lyapunov functional for equation (6):

$$E(w) = \int_{-1}^1 \left( \frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-y^2) + \frac{(p+1)}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy, \quad (8)$$

defined for  $(\partial_s w, w) \in \mathcal{H}$  where

$$\mathcal{H} = \left\{ q \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^1 \left( q_1^2 + (q_1')^2(1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \quad (9)$$

From Proposition 1 in [14], we know that the only stationary solutions of (6) in the space  $\mathcal{H}$  are  $q \equiv 0$  or  $w(y) \equiv \pm \kappa(d, y)$ , where  $d \in (-1, 1)$  and

$$\kappa(d, y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \quad (10)$$

Note that the set of stationary solutions is made of 3 connected components. One wonders whether these stationary solutions are good candidates for the convergence of  $w_{x_0}(y, s)$ . As a matter of fact, that is the case when  $x_0 \in \mathcal{R}$  as we see from the following result (see Corollary 4 in [14] and Theorem 6 in [16]):

**Theorem 3 (Asymptotic behavior near the blow-up graph)**

(i) **Case where  $x_0 \in \mathcal{R}$ : Existence of an asymptotic profile.** *There exist  $\delta_0(x_0) > 0$ ,  $d(x_0) \in (-1, 1)$ ,  $|\theta(x_0)| = 1$ ,  $s_0(x_0) \geq -\log T(x_0)$  such that for all  $s \geq s_0$ :*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0)} \quad (11)$$

for some positive  $\mu_0$  and  $C_0$  independent from  $x_0$ . Moreover,  $E(w_{x_0}(s)) \rightarrow E(\kappa_0)$  as  $s \rightarrow \infty$ .

(ii) **Case where  $x_0 \in \mathcal{S}$ : decomposition into a sum of decoupled solitons.** *It holds that*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(x_0)} e_i^* \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0 \text{ and } E(w_{x_0}(s)) \rightarrow k(x_0)E(\kappa_0) \quad (12)$$

as  $s \rightarrow \infty$ , for some

$$k(x_0) \geq 2, \quad e_i^* = e_1^*(-1)^{i+1}$$

and continuous  $d_i(s) = -\tanh \zeta_i(s) \in (-1, 1)$  for  $i = 1, \dots, k(x_0)$ . Moreover, for some  $C_0 > 0$ , for all  $i = 1, \dots, k(x_0)$  and  $s$  large enough,

$$\left| \zeta_i(s) - \left( i - \frac{(k(x_0) + 1)}{2} \right) \frac{(p-1)}{2} \log s \right| \leq C_0. \quad (13)$$

**Remark:** Note that some elements in the description in similarity variables given above have a geometrical interpretation:

- when  $x_0 \in \mathcal{R}$ , the paramter  $d(x_0)$  appearing in (11) is equal to the slope  $T'(x_0)$  of the blow-up curve (hence  $d(x_0) = T'(x_0)$ );
- when  $x_0 \in \mathcal{S}$ ,  $k(x_0) \geq 2$  is the number of solitons in the decomposition (12) appears also in the upper bound estimate on  $T(x)$  for  $x$  near  $x_0$  given in (4).

**Remark:** The proof of the convergence in (i) has two major difficulties:

- the linearized operator of equation (6) around the profile  $\kappa(d, y)$  is not selfadjoint, which makes the standard tools unefficient for the control of the negative part of the spectrum. Fortunately, the Lyapunov functional structure will be useful in this matter;
- all the non zero stationary solutions of equation (6) are non isolated, which generates a null eigenvalue difficult to control in the linearization of equation (6) around  $\kappa(d, y)$ . A modulation technique is then used to overcome this difficulty.

Extending the definition of  $k(x_0)$  defined for  $x_0 \in \mathcal{S}$  after (12) by setting

$$k(x_0) = 1 \text{ for all } x_0 \in \mathcal{R},$$

we proved the following energy criterion in [16] and using the monotonicity of the Lyapunov functional  $E(w)$ , we have the following consequence from the blow-up behavior in Theorem 3:

**Proposition 2.1 (An energy criterion for non characteristic points; see Corollary 7 in [16])**

(i) For all  $x_0 \in \mathbb{R}$  and  $s_0 \geq -\log T(x_0)$ , we have

$$E(w_{x_0}(s_0)) \geq k(x_0)E(\kappa_0).$$

(ii) If for some  $x_0$  and  $s_0 \geq -\log T(x_0)$ , we have

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then  $x_0 \in \mathcal{R}$ .

### 3 A Liouville theorem and a trapping result near the set of stationary solutions

The following Liouville theorem is crucial in our analysis:

**Theorem 4 (A Liouville Theorem for equation (6))** Consider  $w(y, s)$  a solution to equation (6) defined for all  $(y, s) \in (-\frac{1}{\delta_*}, \frac{1}{\delta_*}) \times \mathbb{R}$  such that for all  $s \in \mathbb{R}$ ,

$$\|w(s)\|_{H^1(-\frac{1}{\delta_*}, \frac{1}{\delta_*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta_*}, \frac{1}{\delta_*})} \leq C^* \quad (14)$$

for some  $\delta_* \in (0, 1)$  and  $C^* > 0$ . Then, either  $w \equiv 0$  or  $w$  can be extended to a function (still denoted by  $w$ ) defined in

$$\{(y, s) \mid -1 - T_0 e^s < d_0 y\} \supset \left(-\frac{1}{\delta_*}, \frac{1}{\delta_*}\right) \times \mathbb{R} \text{ by } w(y, s) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}}, \quad (15)$$

for some  $T_0 \geq T^*$ ,  $d_0 \in [-\delta_*, \delta_*]$  and  $\theta_0 = \pm 1$ , where  $\kappa_0$  defined in (10).

**Remark:** Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \quad (16)$$

where  $u : (x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$ ,  $p > 1$  and  $(N - 2)p < N + 2$ , the blow-up time  $T$  is unique for equation (16). The blow-up set is the subset of  $\mathbb{R}^N$  such that  $u(x, t)$  does not remain bounded as  $(x, t)$  approaches  $(x_0, T)$ . In [17], the second author proved the  $C^2$  regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [10] and [9] was crucially needed for the proof of the regularity result in the heat equation. The above Liouville Theorem is crucial for the regularity of the blow-up set for the wave equation (Theorem 3).

The second fundamental crucial result for our contributions is given by the following trapping result from [14] (See Theorem 3 in [14] and its proof):

**Proposition 3.1 (A trapping result near the sheet  $d \mapsto \kappa(d, y)$  of stationary solutions)** *There exists  $\epsilon_0 > 0$  such that if  $w \in C([s^*, \infty), \mathcal{H})$  for some  $s^* \in \mathbb{R}$  is a solution of equation (6) such that*

$$\forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0) \text{ and } \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^* \quad (17)$$

for some  $d^* = -\tanh \xi^*$ ,  $\omega^* = \pm 1$  and  $\epsilon^* \in (0, \epsilon_0]$ , then there exists  $d_\infty = -\tanh \xi_\infty$  such that

$$|\xi_\infty - \xi^*| \leq C_0 \epsilon^* \text{ and } \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0. \quad (18)$$

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